Survey of Compressive Sensing

Usham Dias, Milind Rane, S. R. Bandewar

Abstract— In the conventional sampling process, for perfect reconstruction of signal according to Nyquist-Shannnon sampling theorem, a band-limited analog signal has to be sampled at atleast twice its highest frequency. The Nyquist-Shannon sampling theorem provides a sufficient condition, but not a necessary one, for perfect reconstruction. The field of compressive sensing provides a stricter sampling condition when the signal is known to be sparse or compressible. Compressive sensing specifically yields a sub-Nyquist sampling criterion. Compressive sensing contains three main problems: sparse representation, measurement matrix and reconstruction algorithm. By now, some available measurement matrices have been discovered, such as Gaussian or Bernoulli independent and identically distributed (i.i.d) random matrices, scrambled Fourier matrix and some structurally random matrices etc. For nonlinear reconstruction, besides the Basis Pursuit (BP) method, several fast greedy algorithms have been proposed, such as the orthogonal matching pursuit (OMP), Regularized OMP, Compressive Sampling OMP. When reconstructing 2D images, besides BP, another popular method is through the minimization of total variation (min-TV) [2].

Index Terms— compressive sensing, sensing matrix, sparse representation, multiwavelet transform;

1 Introduction

WHILE the Nyquist-Shannon sampling theorem states that a certain minimum number of samples is required in order to perfectly capture an arbitrary band-limited signal; but when the signal is sparse in a known basis we can vastly reduce the number of measurements that need to be stored. This is the fundamental idea behind CS: rather than first sampling at a high rate and then compressing the sampled data, we would like to find ways to directly sense the data in a compressed form, at a lower sampling rate. CS differs from classical sampling in three important respects. First, sampling theory typically considers infinite length, continuous-time signals. In contrast, CS is a mathematical theory focused on measuring finite-dimensional vectors in R^N. Second, rather than sampling the signal at specific points in time, CS systems typically acquire measurements in the form of inner products between the signal and more general test functions. Thirdly, the two frameworks differ in the manner in which they deal with signal recovery. In the Nyquist-Shannon framework, signal recovery is achieved through sinc interpolation - a linear process that requires little computation and has a simple interpretation. In CS, however, signal recovery from the compressive measurements is typically achieved using highly nonlinear methods.

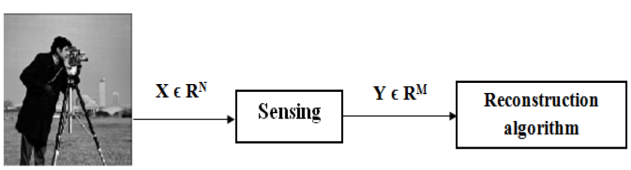

Fig.1: Block diagram for compressive sensing

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2 Compressive Sensing Paradigm

The block diagram for signal processing using compressive sensing is shown in figure 1. The scene under observation is capture using some sensing matrix, which maps the signal from N-dimensional space to M-dimensions, where M<<N. Thus it captures the signal in a compressed form, rather than sampling at Nyquist rate and then compressing. Finally the M-dimensional data needs to be reconstructed back to the N-dimensional space using efficient reconstruction algorithms.

CS theory asserts that one can recover certain signals and images from far fewer measurements M than data samples N. To achieve this CS relies on two principles:

1. *Sparsity /compressibility:* This reflects the fact that the information contained in a signal can be much smaller that it's effective bandwidth. CS exploits explicitly the fact that the data are economically represented in some dictionary ϕ.
2. *Incoherence between the sensing modality and ϕ:* This extends the uncertainty principle between time and frequency in the sense that signals that are sparse in ϕ must be spread out in the domain in which they are acquired; that is, the sensing vectors are as different as possible from the sparsity atoms (and vice versa), and unlike the signal, the sensing vectors must have a dense representation in ϕ.

3 Sparse Representation

In the last decade, sparsity has emerged as one of the leading concepts in a wide range of signal-processing applications (restoration, feature extraction, source separation, and compression, to name only a few applications). Sparsity has long been an attractive theoretical and practical signal property in many areas of applied mathematics.

Recently, researchers spanning a wide range of viewpoints have advocated the use of overcomplete signal representations. Such representations differ from the more traditional representations because they offer a wider range of generating elements (called atoms). Indeed, the attractiveness of redundant signal representations relies on their ability to economically (or compactly) represent a large class of signals. Potentially, this wider range allows more flexibility in signal representation and adaptivity to its morphological content and entails more effectiveness in many signal-processing tasks (restoration, separation, compression, and estimation).

4 SENSING MATRIX

The sensing mechanisms collect information about a signal x(t) by linear functional recordings,

$$y_k = \langle x, a_k \rangle \qquad k = 1, \ldots, M. \qquad (1)$$

That is, we simply correlate the object we wish to acquire with the waveforms ak(t). This is a standard setup. If the sensing waveforms are Dirac delta functions (spikes), for example, then y is a vector of sampled values of x in the time or space domain. If the sensing waveforms are sinusoids, then y is a vector of Fourier coefficients; this is the sensing modality used in magnetic resonance imaging (MRI).

We can represent this process mathematically as

$$y = Ax; \qquad (2)$$

where A is an M x N matrix whose rows are the sensing waveforms ak; and $y \in R^M$. The matrix A represents a dimensionality reduction, i.e., it maps R^N, where N is generally large, into R^M, where M is typically much smaller than N. Note that in the standard CS framework we assume that the measurements are non-adaptive, meaning that the rows of A are fixed in advance and do not depend on the previously acquired measurements. There are two main theoretical questions in CS. First, how should we design the sensing matrix **A** to ensure that it preserves the information in the signal x? Second, how can we recover the original signal x from measurements y? In the case where our data is sparse or compressible, we will see that we can design matrices A with M<< N that ensure that we will be able to recover the original signal accurately and efficiently using a variety of practical algorithms. To recover a unique k -sparse vector (a vector with atmost $k < N$ nonzero entries) x; restrictions are imposed on **A** like satisfying the Null Space Property (NUS), the Restricted Isometry Property (RIP) and or some desired Coherence.

5 RECENT ADVANCES

Currently, researchers always use orthogonal wavelet to represent the images. But the wavelet only has single scaling function and cannot simultaneously satisfy the orthogonality, high vanishing moments, compact support, symmetry characteristic and regularity. Developed from the theory of wavelet, multiwavelet transform, can simultaneously satisfy the five characteristics, and provides a great potential to obtain high performance coding. Paper [7], proposes a compressive sensing image reconstruction based on sparse representation of the image in multi-wavelet transform domain while using Orthogonal Matching Pursuit iterative as the reconstruction algorithm. To evaluate CS reconstruction, they deploy both the OMP and DMWT (discrete multi-wavelet transform) using random Gaussian and Bernoulli measurement and compare the results using 256 x 256 standard gray-scale image. Image reconstruction in compressive sensing using multiwavelet transform and wavelet transform are much better than using discrete cosine transform. Furthermore using multiwavelet domain is better than using wavelet domain. The result of CS reconstruction using Gaussian measurement from M=150, 170, 190 are shown in Table I. According to the comparison in Table I, it is confirmed that image reconstruction for compressive sensing using multi-wavelet and orthogonal matching pursuit is better.

TABLE I: RESULTS USING GAUSSIAN MATRIX

M	DCT transform	Wavelet Transform	Multi-wavelet transform
150	25.3225	28.2376	28.9140
170	26.4840	30.7201	31.1231
190	28.0412	32.6347	32.9589

Comparing PSNR for the image of reconstruction at several measurements M x N (N=256), it was noted that, the more measurements M are taken, the better the quality of image reconstruction it is. The result of CS reconstruction using Bernoulli measurement from M=150, 170, 190 are shown as follows:

TABLE II: RESULTS USING BERNOULLI MATRIX

M	DCT transform	Wavelet Transform	Multi-Wavelet transform
150	24.6866	28.3624	29.0561
170	26.5264	30.4853	31.2575
190	28.0339	32.5860	33.2606

Most of existing methods for CS image reconstruction are suitable for piecewise smooth image, but do not behave well on texture-rich natural image. In paper [2], a new optimization problem for CS image reconstruction is proposed, in which different regularization terms are introduced for different morphological components of image. Experimental results show that the proposed method can be applied to reconstruct texture-rich images besides piecewise smooth ones, and outperforms the existing methods on preserving detail features. For nonlinear reconstruction methods, besides the BP method several fast greedy algorithms have been proposed, such as the orthogonal matching pursuit (OMP), the tree-based OMP and the stage-wise orthogonal matching pursuit (StOMP). Other algorithms include iterative soft-thresholding and projection onto convex sets, etc. When reconstructing 2D images, besides BP, another popular method is through the minimization of total variation (min-TV). The min-TV method assumes that the image is gradient sparse and usually offers reconstructed images with better visual quality. Unfortunately, these two methods and other relevant techniques are not appropriate to reconstruct texture-rich natural images. The texture feature is often smoothed during the reconstruction process. Paper [2], proposes a new optimization problem for image reconstruction by taking advantages of morphological component analysis (MCA), which can preserve texture feature while not degrading the visual quality of whole image as shown in table III.

In paper [8], CS reconstruction problem of images are discussed from a multivariate point of view. Most conventional wavelet-based CS reconstruction methods assume that the wavelet coefficients are mutually independent. However, significant statistical dependency exists among the wavelet coefficients of images. The statistical structures of the wavelet coefficients are considered for CS reconstruction of images that are sparse or compressive in wavelet domain. A multivariate pursuit algorithm (MPA) based on the multivariate models is developed. Several multivariate scale mixture models are used as the prior distributions of MPA. Their method reconstructs the images by means of modelling the statistical dependencies of the wavelet coefficients in a neighbourhood. Multivariate algorithms proposed here present superior performance compared with some state-of-the-art CS algorithms.

In CS applications, the acquisition of the linear projections *Ax* requires a physical implementation. In most cases, the use of an i.i.d. Gaussian random matrix A is either impossible or overly expensive. This motivates the study of easily implementable CS matrices. Two such matrices are the Toeplitz and circulant matrices [9], which have been shown to be almost as effective as the Gaussian random matrix for CS encoding/decoding. In toeplitz matrix, every left-to-right descending diagonal is constant, i.e., $T_{i,j} = T_{i+1,j+1}$. If T satisfies the additional property that $t_i = t_{n+i}$; for all i where n is the number of columns, it is also a circulant matrix C. An m x n general matrix has mn degrees of freedom, but a partial circulant matrix of the same size has at most n degrees of freedom. Hence, a random circulant matrix is generated from much fewer independent random numbers or is much less random than an i.i.d. random matrix of the same size. This fact seemingly suggests that a random circulant matrix would yield less incoherent projections, and consequently worse CS recovery. However, numerical results in [9] show that circulant matrices can be equally effective as i.i.d. random matrices.

In paper [10], they investigate signal recovery procedure for the case where A is binary and very sparse. They show that, both in theory and in practice, sparse matrices are essentially as "good" as the dense ones. At the same time, sparse binary matrices provide additional benefits, such as reduced encoding and decoding time. They also report experimental results which indicate that, in practice, binary sparse matrices are as "good" as random Gaussian or Fourier matrices when used in Linear programming(LP) decoding (both in terms of the number of necessary measurements, and in terms of the recovery error). At the same time, the LP decoding is noticeably faster for sparse matrices.

6 CONCLUSION

Compressive sensing has changed the way the intellectual community deals with signals especially its acquisition and compression. It provides a necessary condition for sampling and faithful reconstruction of signal, whose bounds are much lower than the conventional Nyquist sampling theorem. The main challenges in compressive sensing are to select the best sparse representation for the signal, measurement matrix for acquisition and algorithm for reconstruction. Research is moving from the conventional DFT, DCT and wavelet sparse domains to multi-wavelet, contourlet, curvelet, ridgelets and bandlets. Gaussian and Bernoulli matrices were the first few sensing matrices discovered. But they are inefficient for hardware implementation. Therefore the intellectual community is looking for sparse matrices as potential candidates to replace the dense random matrices. Toeplitz and circulant matrices are also being used as measurement matrix. Basis pursuit and orthogonal matching pursuit (OMP) has proved efficient for reconstruction of sparse signals. Other algorithms which can

TABLE III: COMPARISON OF THREE RECONSTRUCTION METHODS (PSNR IN DB)

Image	Boat		Lena		Barbara	
M	15000	25000	15000	25000	15000	25000
BP	23.93	27.96	23.87	28.01	22.60	25.70
Min-TV	29.67	33.41	29.69	33.18	25.67	28.50
Method in [2]	30.96	35.39	30.84	35.29	26.86	29.75

be considered are the Tree-Based OMP, stage-wise OMP and compressive sampling matching pursuit (CoSAMP).

ACKNOWLEDGMENT

We would like to extend our sincere thanks to Prof. Milind Kamble, Dept. Of Electronics & Telecommunication, Vishwakarma Institute of Technology, for his help in the field of optimization.

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